Quadratic equations. Explanatory notes.\textsuperscript{1}

1. Quadratic equations

An equation of the form

\[ ax^2 + bx + c = 0, \]

where \( a, b, c \) are some real numbers, and \( a \neq 0 \), is called a quadratic equation.

Equations of the form \( ax^2 + bx = 0 \) and \( ax^2 + c = 0 \) with \( a \neq 0 \) are called incomplete quadratic equations.

Equation \( ax^2 + bx = 0 \) can be transformed to the form: \( x(ax + b) = 0 \), whence it follows, that the solutions of the obtained equation are the numbers \( x = 0 \) and \( x = -b/a \).

Equation \( ax^2 + c = 0 \) with \( a \neq 0 \) is equivalent to the equation \( x^2 + c/a = 0 \). Whence it follows, that if \( c = 0 \) the equation has unique solution \( x = 0 \). If \( c/a > 0 \), then the equation does not have any real solutions, as \( x^2 + c/a \geq c/a \), i.e. for any \( x \) the left hand side of the equation differs from zero. If \( c/a < 0 \), then the equation can be transformed to the form:

\[
\left( x + \sqrt{\frac{-c}{a}} \right) \left( x - \sqrt{\frac{-c}{a}} \right) = 0,
\]

whence it follows, that the equation has two solutions, \( x_1 = \sqrt{-\frac{c}{a}} \) and \( x_2 = -\sqrt{-\frac{c}{a}} \).

Example 1. Solve the equations:

\begin{align*}
\text{a)} & \ 3x^2 = 0, \quad \text{b)} \ 4x^2 - 3x = 0, \quad \text{c)} \ 5x^2 + 2 = 0, \quad \text{d)} \ 4x^2 - 9 = 0. \\
\end{align*}

Solutions.

\textbf{a)} Equation has the unique solution \( x = 0 \).

\textbf{b)} Equation can be transformed to the equivalent form \( x(4x - 3) = 0 \), whence it follows that it has two solutions \( x_1 = 0 \) and \( x_2 = 3/4 \).

\textbf{c)} Equation does not have any real solutions, as the left hand side of the equation, for any real value of \( x \), is larger or equal to 2.

\textbf{d)} We transform the equation to the form \( (2x - 3)(2x + 3) = 0 \), whence it follows that the equation has two solutions \( x_1 = 3/2 \) and \( x_2 = -3/2 \).

Now we consider equation (1), where numbers \( a, b, c \) differ from zero. We transform the left hand side of this equation in the following manner:

\[
ax^2 + bx + c = a \left( x^2 + \frac{b}{a} x + \frac{c}{a} \right) =
\]

\[= a \left( x^2 + 2 \frac{b}{2a} x + \left( \frac{b}{2a} \right)^2 - \left( \frac{b}{2a} \right)^2 + \frac{c}{a} \right) =
\]

\[= a \left( \left( x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a^2} \right).
\]

\textsuperscript{1}preparing this material we used an old Russian source from the Correspondence Physicotechnical School at MFTI (ZFTSH), Moscow, Russia
Expression $b^2 - 4ac$ is called discriminant of the quadratic equation (1) and denoted by letter $D$.

- If $D \geq 0$, then expression (2) can be factored

$$a \left( x + \frac{b + \sqrt{D}}{2a} \right) \left( x + \frac{b - \sqrt{D}}{2a} \right) = 0.$$  

We introduce notations:

$$x_1 = \frac{-b + \sqrt{D}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{D}}{2a}.$$

Then equation (1) can be transformed to the form

$$a(x - x_1)(x - x_2) = 0,$$

where the numbers $x_1$ and $x_2$ are zeros of equation (1).

Formulae (3) for the solutions of equation (1) are usually written in one formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ or equivalently } x_{1,2} = \frac{-b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}.$$ 

- If $D = 0$, then $x_1 = x_2$, i.e. the solutions coincide, and equation (1) is reduced to $a(x - x_1)^2 = 0$.

- If $D < 0$, then

$$\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2} \geq -\frac{D}{4a^2} > 0$$

for any $x$. Thus in this case equation (1) does not have any real solutions.

**Example 2.** Solve the following quadratic equations:

- a) $2x^2 + 3x - 5 = 0$,  
- b) $3x^2 + 4x + 2 = 0$,  
- c) $9x^2 - 6x + 1 = 0$,  
- d) $\sqrt{2} \cdot x^2 - \sqrt{3} \cdot x - \sqrt{2} = 0$.

**Solutions.**

a) First we find the discriminant of the given quadratic equation:

$$D = 3^2 - 4 \cdot 2(-5) = 9 + 40 = 49.$$ 

As $D = 49 > 0$, then according to formula (5) we get:

$$x_{1,2} = \frac{-3 \pm 7}{4} = \frac{-3 \pm 7}{4}.$$ 

Thus, the equation has two zeros $x_1 = 1$ and $x_2 = -5/2$.

b) As $D = 4^2 - 4 \cdot 3 \cdot 2 = 16 - 24 = -8 < 0$, then the equation does not have solutions in real numbers.

c) As $D = 6^2 - 4 \cdot 9 \cdot 1 = 0$, then the equation has the unique zero $x = 16/18 = 1/3$.

d) $D = 3 - 4 \cdot \sqrt{2}(-\sqrt{2}) = 11$ and the roots are

$$x_1 = \frac{\sqrt{3} + \sqrt{11}}{2\sqrt{2}} \text{ and } x_2 = \frac{\sqrt{3} - \sqrt{11}}{2\sqrt{2}}.$$ 

If in equation (1) the number $b = 2b_1$, then formula (5) reads

$$x_{1,2} = \frac{-b_1 \pm \sqrt{b_1^2 - ac}}{a} \text{ or equivalently } x_{1,2} = \frac{-b_1}{a} \pm \sqrt{\left(\frac{b_1}{a}\right)^2 - \frac{c}{a}}.$$}

In formula (6) the number $b_1$ is equal half of the coefficient for $x$ in equation (1).
We denote expression $b_1^2 - ac$ by $D_1$. Consequently, solutions of quadratic equation $x^2 + 2b_1x + c = 0$ are given by formula

$$x_{1,2} = \frac{-b_1 \pm \sqrt{D_1}}{a} \text{ if } D_1 = b_1^2 - ac \geq 0.$$

**Example 3. Solve the quadratic equations:**

- a) $3x^2 - 4x - 1 = 0$,
- b) $2x^2 + 2x + 5 = 0$.

**Solutions.**

- a) $D_1 = 2^2 - 3(-1) = 7 > 0$. By formula (6) we have:
  $$x_{1,2} = \frac{2 \pm \sqrt{7}}{3}, \text{ i.e. } x_1 = \frac{2 + \sqrt{7}}{3}, \ x_2 = \frac{2 - \sqrt{7}}{3}.$$  

- b) $D_1 = 1^2 - 2 \cdot 5 = -9 < 0$. The equation does not have real solution.

**Example 4. Find, which of the equations given below are equivalent:**

- a) $6x^2 + x - 1 = 0$ and $(x + \frac{1}{2})(x - \frac{1}{3}) = 0$
- b) $2x - 6 = 0$ and $x^2 - 6x + 9 = 0$
- c) $x^2 + x + 1 = 0$ and $x^2 - x + 1 = 0$
- d) $x + 1 = 0$ and $2x^2 + x - 1 = 0$.

**Solutions.**

- a) The first equation has two solutions
  $$x_{1,2} = \frac{-1 \pm \sqrt{25}}{12} = \frac{-1 \pm 5}{12}, \ \ x_1 = \frac{-1 + 5}{12} \text{ and } x_2 = \frac{-1 - 5}{12}. $$

  These and only these numbers are also zeros of the second equation, thus, the equations are equivalent.

- b) The first equation has unique solution $x = 3$. The second equation transforms in $(x-3)^2 = 0$, i.e. also only has one solution $x = 3$. Thus the equations are equivalent.

- c) For both equations the discriminant is equal to $1 - 4 = -3 < 0$, consequently both equations do not have any real solutions and thus the equations are equivalent.

- d) The first equation has one solution $x = -1$, while the second equation has two solutions:
  $$x_1 = \frac{-1 + 3}{4} = \frac{1}{2} \text{ and } x_2 = \frac{-1 - 3}{4} = -1. $$

  The number $1/2$ is solution to the second equation but is not solution to the first equation. Consequently, the equations are not equivalent.

**Example 5. Find all prime numbers $p$ and $q$ such that the equation $x^2 - px - q = 0$ has a solution which is a prime number.**

**Solution.** Suppose a prime number $x = n$ is a solution, then we have $n^2 - pn - q = 0$, whence it follows that $q = n(n - p)$. As $q$ is a prime number and $n$ is prime number, then $n - p = 1$, i.e. $n = p + 1$ and $q = p + 1$. The number $p$ can be equal only 2, as in any other case the number $p + 1$ would be even and then the number $q$ could not be a prime number. It follows that the desired quadratic equation has the form: $x^2 - 2x - 3 = 0$.

2. The Vieta theorem. The reduced quadratic equation

Let us find the sum and the product of the zeros of the quadratic equation (1). From formula (3) it follows $x_1 + x_2 = -b/a$,

$$x_1 \cdot x_2 = \frac{(-b + \sqrt{D})(-b - \sqrt{D})}{4a^2} = \frac{b^2 - D}{4a^2} = \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{c}{a}.$$
Whence it follows the statement which is called the Vieta theorem:

**if the solutions of the quadratic equation** \( ax^2 + bx + c = 0 \) **exist, then the sum of the zeros of the quadratic equation is equal to** \(-b/a\), **and their product is equal to** \( c/a\).  

For example, solutions of the quadratic equation \( 2x^2 - 3x - 5 = 0 \) exist as \( D = 9 + 4 \cdot 2 \cdot 5 = 49 > 0 \). According to the Vieta theorem, the sum of the zeros of this equation is equal to \( 3/2 \) and their product is \(-5/2\).

**Example 6.** Without solving the quadratic equation, find the sum of the squares of solutions the quadratic equation \( ax^2 + bx + c = 0 \), where \( a \neq 0 \), \( b^2 - 4ac > 0 \).

**Solution.** It follows from the Vieta theorem that \( x_1 + x_2 = -b/a \) and \( x_1 \cdot x_2 = c/a \). We transform expression \( x_1^2 + x_2^2 \):

\[
x_1^2 + x_2^2 = x_1^2 + x_2^2 + 2x_1x_2 - 2x_1x_2 = (x_1 + x_2)^2 - 2x_1x_2.
\]

This gives us

\[
x_1^2 + x_2^2 = \left( -\frac{b}{a} \right)^2 - 2\frac{c}{a} = \frac{b^2 - 2ac}{a^2}.
\]

Equation \( x^2 + px + q = 0 \) is called a **reduced quadratic equation**. In this equation, the coefficient of \( x^2 \) is equal to 1. The formula for the zeros for the reduced quadratic equation takes the form

\[
x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q} \\
\text{or } x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{(p/2)^2}{4} - q}.
\]

The Vieta theorem for the reduced quadratic equation reads: if \( x_1 \) and \( x_2 \) are solutions of the reduced quadratic equation, then \( x_1 + x_2 = -p \), \( x_1 \cdot x_2 = q \).

**Inverse Vieta theorem:** If \( p\), \( q\), \( x_1 \) and \( x_2 \) are such that \( x_1 + x_2 = -p \), \( x_1 \cdot x_2 = q \), then \( x_1 \) and \( x_2 \) are the zeros of equation \( x^2 + px + q = 0 \).

For the proof, substitute in the equation \( x^2 + px + q = 0 \) the expression \(-(x_1 + x_2)\) instead of \( p \) and the expression \( x_1 \cdot x_2 \) instead of \( q \). Then we get

\[
x^2 - (x_1 + x_2)x + x_1 \cdot x_2 = x^2 - x_1x - x_2x + x_1x_2 = (x - x_1)(x - x_2).
\]

As a result, it follows that the numbers \( x_1 \) and \( x_2 \) are solutions of the equation \( x^2 + px + q = 0 \).

**Example 7.** Construct the reduced quadratic equation with the roots 1/2 and 3/7.

**Solution.** It follows from the inverse Vieta theorem that the given numbers are zeros of the reduced quadratic equation

\[
x^2 - \left( \frac{1}{2} + \frac{3}{7} \right)x + \frac{1}{2} \cdot \frac{3}{7} = 0,
\]

i.e. equation

\[
x^2 - \frac{13}{14}x + \frac{3}{14} = 0.
\]

Observe that the given numbers are also solutions of the quadratic equation \( 14x^2 - 13x + 3 = 0 \) which follows from the previous equation by multiplication with 14.

**Example 8.** The zeros \( x_1 \) and \( x_2 \) of the quadratic equation \( x^2 + 6x + q = 0 \) satisfy \( x_2 = 2x_1 \). Find \( q\), \( x_1 \) and \( x_2 \).

**Solution.** From the Vieta theorem it follows that \( x_1 + x_2 = 3x_1 = -6 \), i.e. \( x_1 = -2 \), and \( x_2 = 2x_1 = -4 \). Then \( q = x_1 \cdot x_2 = 8 \).
Example 9. Denote $x_1$, $x_2$ the zeros of the quadratic equation $2x^2 - 3x - 9 = 0$. Without solving the equation find $\frac{x_2}{1 + x_1} + \frac{x_1}{1 + x_2}$.

**Solution.** We transform the expression:

$$\frac{x_2}{1 + x_1} + \frac{x_1}{1 + x_2} = \frac{(x_2 + x_1) + x_2^2 + x_1^2}{1 + (x_1 + x_2) + x_1 \cdot x_2} = \frac{(x_2 + x_1) + (x_1 + x_2)^2 - 2x_1x_2}{1 + (x_1 + x_2) + x_1 \cdot x_2}.$$  

By the Vieta theorem $x_1 + x_2 = 3/2$ and $x_1 \cdot x_2 = -9/2$, thus we have

$$\frac{\frac{3}{2} + \left(\frac{3}{2}\right)^2 - 2 \left(-\frac{9}{2}\right)}{1 + \frac{3}{2} - \frac{9}{2}} = \frac{\frac{3}{2} + \frac{9}{2} + 9}{-2} = \frac{51}{8}.$$  

Example 10. Let $x_1$ and $x_2$ are solutions of the quadratic equation $x^2 + 13x - 17 = 0$. Construct the quadratic equation with zeros $2 - x_1$ and $2 - x_2$.

**Solution.** By the Vieta theorem $x_1 + x_2 = -13$ and $x_1 \cdot x_2 = -17$. The sum of the numbers $2 - x_1$ and $2 - x_2$ is equal to $4 - (x_1 + x_2) = 4 + 13 = 17$, the product of these numbers is equal to $(2 - x_1)(2 - x_2) = 4 - 2(x_1 + x_2) + x_1 x_2 = 4 - 2(-13) - 17 = 13$. By using the inverse Vieta theorem we get the quadratic equation $x^2 - 17x + 13 = 0$, with the given zeros.

3. Solution of equations which can be reduced to the quadratic equations

Equation

$$ax^4 + bx^2 + c = 0,$$

where $a, b, c$ are some real numbers and $a \neq 0$, is called a *biquadratic equation*. By the change of variable $u = x^2$, this equation reduces to quadratic equation $au^2 + bu + c = 0$.

Example 11. Solve biquadratic equations:

a) $2x^4 - 3x^2 + 1 = 0$,  b) $5x^4 - 7x^2 - 6 = 0$,  c) $7x^4 + 9x^2 + 2 = 0$.

**Solution.**

a) By the change of variable $u = x^2$, we get quadratic equation $2u^2 - 3u + 1 = 0$. By the formula for the solutions of quadratic equation we get $u = (3 \pm \sqrt{9 - 8})/4 = (3 \pm 1)/4$, i.e. $u_1 = 1$, $u_2 = 1/2$. Whence it follows that $x^2 = 1$ or $x^2 = 1/2$, and thus the given equation has 4 solutions: $x_1 = 1$, $x_2 = -1$, $x_3 = \sqrt{2}$ and $x_4 = -\sqrt{2}$.

b) After the change $u = x^2$ we get the equation $5u^2 - 7u - 6 = 0$. We find its zeros $u_{1,2} = (7 \pm \sqrt{49 + 4 \cdot 5 \cdot 6})/10 = 7 \pm 13/10$ and thus $u_1 = 2$ and $u_2 = -3/5$. Equation $x^2 = 2$ has two solutions: $x_1 = \sqrt{2}$ and $x_2 = -\sqrt{2}$. Equation $x^2 = -3/5$ does not have real solutions. Hence, the given biquadratic equation has two real solutions $\sqrt{2}$ and $-\sqrt{2}$.

c) Equation does not have any real solutions, as $7x^4 + 9x^2 + 2 \geq 2$ for any $x \in \mathbb{R}$.

Example 12. Solve the equation

$$\frac{2x + 1}{x - 1} + \frac{x + 1}{2x + 1} = \frac{5x + 4}{(x - 1)(2x + 1)}.$$

A common divisor for the denominators of the fractions in the given equation is equal to $(x - 1)(2x + 1)$. Multiplying both sides of the equation with $(x - 1)(2x + 1)$, we get new equation

$$(2x + 1)^2 + (x + 1)(x - 1) = 5x + 4, \ 4x^2 + 4x + 1 + x^2 - 1 = 5x + 4, \ 5x^2 - x - 4 = 0.$$

The solutions of the obtained quadratic equation are

$$x_{1,2} = \frac{1 \pm \sqrt{1 + 80}}{10} = \frac{1 \pm 9}{10},$$
\( x_1 = 1, x_2 = -4/5. \)

Note that the transformation made was not an equivalent one. Namely, by multiplying the initial equation by \((x-1)/(2x+1)\) we could introduce two additional solution to the equation. Those one where the multiplier is equal to zero.

For \( x = 1 \), both sides of the initial equation are not defined and thus this number is not a solution of the equation. For \( x = -4/5 \) the least common divisor is not equal to zero and hence this number is the solution of the given equation.

**Example 13.** Solve the equation \((x + 2)^2 + \frac{24}{x^2 + 4x} = 18.\)

**Solution.** Introduce the new variable \( t = (x + 2)^2 \). As \( x^2 + 4x = x^2 + 4x + 4 - 4 \), then we have \( x^2 + 4x = t - 4 \), and with respect to \( t \), we get the equation \( t + 24/(t - 4) = 18 \). Multiply both sides of the last equation with \( t - 4 \) and get
\[
t^2 - 4t + 24 = 18t - 72, \quad t^2 - 22t + 96 = 0.
\]

The solutions of this quadratic equation are 6 and 16. Now we solve \((x + 2)^2 = 16\), whence it follows that \( x + 2 = \pm 4 \), i.e. \( x_1 = 2 \) and \( x_2 = -6 \). Next solve the equation \((x + 2)^2 = 6\), whence it follows that \( x_3 = -2 + \sqrt{6} \) and \( x_4 = -2 - \sqrt{6} \). Motivate that all found numbers are solutions of the initial equation.

**Example 14.** Solve the equation \( \frac{x^2 + 2x + 7}{x^2 + 2x + 3} = 4 + 2x + x^2 \).

**Solution.** Introduce the new variable \( x^2 + 2x + 3 = t \), then with respect to \( t \) we have the equation \((t + 4)/t = t + 1 \). Multiply both sides of this equation with \( t \) to get \( t + 4 = t^2 + t \), \( t^2 = 4 \), \( t_1 = 2 \), \( t_2 = -2 \). We solve equation \( x^2 + 2x + 3 = 2 \), \( x^2 + 2x + 1 = 0 \). It has the only solution \( x = -1 \). Equation \( x^2 + 2x + 3 = -2 \), i.e. \( x^2 + 2x + 5 = 0 \), does not have solutions. Hence, the original equation has the only solution \( x = -1 \).

**Example 15.** Solve the equation \( x^2 + \frac{81x^2}{(9 + x)^2} = 40 \).

**Solution.** The left hand side of the equation is the sum of the squares of \( x \) and \( 9x/(9 + x) \). We add \( -2x \cdot 9x/(9 + x) \) to the both sides of the equation and get:
\[
\left( x - \frac{9x}{9 + x} \right)^2 = 40 - 2x \cdot \frac{9x}{9 + x}, \quad \left( \frac{x^2}{9 + x} \right)^2 = 40 - \frac{18x^2}{9 + x}.
\]

Introduce the new variable \( t = x^2/(9 + x) \). With respect to \( t \) the equation is \( t^2 + 18t - 40 = 0 \). Its zeros are the numbers 2 and -20.

For \( t = 2 \), the equation with respect to \( x \) is \( x^2/(9 + x) = 2 \), which after multiplication of both sides by \( 9 + x \) reduces to the quadratic equation \( x^2 - 2x - 18 = 0 \). Its two zeros are \( 1 \pm \sqrt{19} \) which simultaneously are solutions to the original equation.

For \( t = -20 \), the equation with respect to \( x \) is \( x^2/(9 + x) = -20 \) or \( x^2 + 20x + 180 = 0 \) which does not have any real solutions.