Observer Forms for Perspective Systems ⋆

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Abstract: Estimation of 3D position information from 2D images in computer vision systems can be formulated as a state estimation problem for a nonlinear perspective dynamic system. The multi-output state estimation problem has been treated by several authors using methods for nonlinear observer design. This paper shows that a perspective system can be transformed to two observer forms, and provides constructive methods for arriving at the transformations. These observer forms lead to straightforward observer designs. First, it is shown that using an output transformation, the system admits an observer form which leads to an observer with linear error dynamics. A second observer design is based on a time scaled block triangular form. Both designs assume a commonly used observability condition. The designs are demonstrated in simulation.

1. INTRODUCTION

The problem of estimating 3D structure and motion from 2D perspective observations can be formulated using a nonlinear perspective dynamic system. The perspective system is obtained by considering the relative motion between a perspective camera and an observed object. The estimation of both structure and motion can be achieved by an observer for states and parameters. Existing approaches have used the extended Kalman filter Azarbayejani and Pentland [1995], Soatto et al. [1996] or adaptive observers Chen and Kano [2004], Dahl et al. [2007b]. The problem of estimating structure when the motion parameters are measured or otherwise assumed available, has been considered using observer-based approaches in Matthies et al. [1989], Jankovic and Ghosh [1995], Matveev et al. [2000], Chen and Kano [2002], Dixon et al. [2003], Dahl et al. [2005], Abdursul et al. [2004], Ma et al. [2005], Karagiannis and Astolfi [2005], Gupta et al. [2006], Martino et al. [2006].

This paper presents structure estimation results, showing how a perspective system can be transformed into two observer forms. These forms naturally lead to observers with simple error dynamics systems. The simplicity of the error dynamics leads to a straightforward stability analysis. Relative to existing related work, the results here show that it is possible to achieve linear time-invariant error dynamics *without any constraints on the type of motion* when an Observer Form (OF) with output transformation is considered, Krener and Respondek [1985]. Previous work in Dahl et al. [2007a] considered the OF without output transformation, and required a constraint on the type of motion which potentially limited the application of the approach. A second contribution of the paper is to demonstrate the application of a Time-Scaled Block Triangular Observer Form (TBTOF) which was first introduced in Wang and Lynch [2006b]. The TBTOF is a generalization of OF and can therefore be applied to a wider class of systems.

Perspective dynamic systems, their observability, and classical OF existence conditions are introduced in Section 2. Section 3 presents the OF and TBTOF, the method of construction for these coordinates, and related observer designs. Simulations are presented in Section 4 and conclusions are drawn in Section 5.

2. BACKGROUND

2.1 Perspective dynamic systems

A perspective dynamic system with three states and two outputs, derived assuming a calibrated pinhole camera and observations of feature points on a rigid object, can be written as e.g. Abdursul et al. [2004], Ma et al. [2004]:

$$\dot{x} = Ax + b, \quad y = \begin{pmatrix} \frac{x_1}{x_3} & \frac{x_2}{x_3} \end{pmatrix}^T \tag{1}$$

with

$$A = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad , \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{2}$$

where we assume $\omega_i, b_i, 1 \le i \le 3$ are constant.

As in e.g. Chen and Kano [2002], a useful alternative formulation of the perspective dynamic system (1) can be obtained by applying an initial change of coordinates

$$\xi = (\xi_1 \ \xi_2 \ \xi_3)^T = \begin{pmatrix} \frac{x_1}{x_3} & \frac{x_2}{x_3} & \frac{1}{x_3} \end{pmatrix}^T \tag{3}$$

which results in

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$$
\begin{aligned}
&\dot{\xi}_1 = -\omega_1\xi_1\xi_2 + \omega_2(1+\xi_1^2) - \omega_3\xi_2 + (b_1 - b_3\xi_1)\xi_3 \\
&\dot{\xi}_2 = \omega_2\xi_1\xi_2 - \omega_1(1+\xi_2^2) + \omega_3\xi_1 + (b_2 - b_3\xi_2)\xi_3 \\
&\dot{\xi}_3 = -(\omega_1\xi_2 - \omega_2\xi_1 + b_3\xi_3)\xi_3 \\
&y_1 = \xi_1, \quad y_2 = \xi_2
\end{aligned} \tag{4}
$$

where the nonlinear terms now occur in the state equations, and the output equations are linear.

2.2 Observability

We use the notation $L_f h(x)$ for the Lie derivative of a function $h(x)$ along a vector field $f(x)$ and the notation $L_f^k h(x)$ for the k times repeated Lie derivative, together with the notation $d\lambda(x)$ for the gradient of a function $\lambda(x)$. Given two vector fields $f(x)$ and $g(x)$, we use the notation $ad_f g$ for the Lie bracket $[f,g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$ and the notation $ad_f^i g$ for repeated Lie bracket $ad_f^i g = [f, ad_f^{i-1} g]$ and $ad_f^0 g = g$.

From Marino and Tomei [1995], Krener and Respondek [1985], a dynamic system

$$\dot{x} = f(x), \quad y = h(x) \tag{5}$$

where $x \in \mathbb{R}^n$ is the state, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^∞ vector field, and $h : \mathbb{R}^n \to \mathbb{R}^s$ is a C^∞ output function, is locally observable in the neighborhood of x_0 if

$$\text{Rank}\left\{dL_f^k h_i(x_0) : 0 \le k \le k_i - 1,\ 1 \le i \le s\right\} = n \tag{6}$$

where $k_i, 1 \le i \le k$ is a set of observability indices. To investigate the observability of the system (4), we verify the observability indices are $\{2,1\}$, and compute the matrix

$$\Omega^s = \begin{pmatrix} dh_1(\xi) \\ dh_2(\xi) \\ dL_f h_1(\xi) \\ dL_f h_2(\xi) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \Omega_{31}^s & \Omega_{32}^s & b_1 - b_3\xi_1 \\ \Omega_{41}^s & \Omega_{42}^s & b_2 - b_3\xi_2 \end{pmatrix} \tag{7}$$

where Ω_{ij}^s, $i \in \{3,4\}$, $j \in \{1,2\}$ are some functions of ξ. According to the observability definition (6), system (4) is locally observable at $\xi = (\xi_1, \xi_2, \xi_3)^T$ if and only if either $\text{Rank}\{dh_1, dL_f h_1, dh_2\} = 3$ or $\text{Rank}\{dh_1, dh_2, dL_f h_2\} = 3$. This implies that the system (4) is observable at ξ if and only if either $b_1 - b_3\xi_1 \ne 0$ or $b_2 - b_3\xi_2 \ne 0$, and accordingly, that the system (1) is observable at $x = (x_1\ x_2\ x_3)^T$ if and only if either $b_1 x_3 - b_3 x_1 \ne 0$ or $b_2 x_3 - b_3 x_2 \ne 0$. The observability condition can be summarized as

$$(b_1 - b_3\xi_1)^2 + (b_2 - b_3\xi_2)^2 \ne 0 \tag{8}$$

which is a commonly obtained expression, referred to as the *focus of expansion* e.g. Chen and Kano [2002], Dixon et al. [2003], Karagiannis and Astolfi [2005] where (8) is required for observer convergence. Without loss of generality, we assume $b_1 - b_3\xi_1 \ne 0$ in this paper. This ensures the perspective system (4) is locally observable in the neighborhood of some $\xi_0 \in \mathbb{R}^3$ with observability indices $k_1 = 2, k_2 = 1$ relative to the outputs $y_1 = \xi_1, y_2 = \xi_2$.

2.3 Observer forms

Given the dynamic system (5), the existence conditions for a change of state coordinates under which the system (5) admits an OF are well-established, Krener and Respondek [1985], Marino and Tomei [1995], Xia and Gao [1989]. System (4) is transformable to OF by a state transformation $z = \Phi(\xi)$ if and only if the following three conditions are fulfilled:

1. The matrices R_j^l and R_j^r

$$
\begin{aligned}
R_j^l &= \{dL_f^k h_i : 0 \le k \le k_j - 1, i \ne j, \\
&\qquad 1 \le i \le 2, dL_f^k h_j : 0 \le k \le k_j - 2)\} \\
R_j^r &= \{dL_f^k h_i : 0 \le k \le \min(k_i, k_j) - 1, i \ne j, \\
&\qquad 1 \le i \le 2, dL_f^k h_j : 0 \le k \le k_j - 2\}
\end{aligned} \tag{9}
$$

have the same rank for all $j, 1 \le j \le 2$.

2. There exist vector fields r_i, $1 \le i \le 2$, such that

$$
\begin{aligned}
&L_{r_i} L_f^{k-1} h_j = \delta_{i,j} \cdot \delta_{k,k_j}, \\
&1 \le i \le 2, \quad 1 \le k \le k_i, \quad 1 \le j \le 2
\end{aligned} \tag{10}
$$

where $\delta_{i,j} = 1$ when $i = j$ and zero otherwise.

3.

$$
\begin{aligned}
&[ad_{-f}^k r_i, ad_{-f}^l r_j] = 0, \\
&1 \le i, j, \le 2, \quad 0 \le k \le k_i - 1, \quad 0 \le l \le k_j - 1
\end{aligned} \tag{11}
$$

Without output transformation, the system (4) admits an OF under the constraint $b_2 = b_3 = 0$ Dahl et al. [2007a] given the observability assumption $b_1 - b_3\xi_1 \ne 0$. In this paper we provide two results which extend the work in Dahl et al. [2007a]. The first result shows the existence of an output transformation $\bar{y} = \Psi(y)$ and a state transformation $z = \Phi(\xi)$ such that (4) is transformable to OF without motion constraints. The second result demonstrates the existence of a TBTOF which provides coordinates allowing for a straightforward observer design, albeit with the same constraint on the motion which appeared in Dahl et al. [2007a] for the dynamic error linearization.

3. OBSERVER FORMS FOR PERSPECTIVE SYSTEMS

This section presents our main results regarding the transformation of (4) to observer forms. We follow two approaches to derive the output transformation $\bar{y} = \Psi(y)$, and give the state transformation $z = \Phi(\xi)$ to observer form. In addition, subsection 3.3 demonstrates how a transformation involving time scaling can be performed, such that the system expressed in the new time scale is transformable to a block triangular observer form. The results have been derived by using a Maple library for observer error linearization Dahl [2008].

3.1 Observer form

An OF for the perspective system in ξ-coordinates (4) can be derived by first finding an output transformation, and then computing a state transformation. An initial observation is that the rank condition (9) is in general not satisfied. This can be seen from the matrices R_j^l, R_j^r for the first output, i.e. $j = 1$,

$$R_1^l = \begin{pmatrix} dh_1 \\ dh_2 \\ dL_f h_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & b_2 - b_3\xi_2 \end{pmatrix}$$

$$R_1^r = \begin{pmatrix} dh_1 \\ dh_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which have different rank unless $b_2 - b_3\xi_2 = 0$. Given that no output transformation is employed, the rank condition can be satisfied when $b_2 = b_3 = 0$, a condition which is used in Dahl et al. [2007a] to derive an observer form for the perspective system (4). The ranks of R_1^l, R_1^r can be made equal if an output transformation

$$\bar{y}_1 = \xi_1, \quad \bar{y}_2 = \psi_2(\xi_1, \xi_2) \tag{12}$$

is used. This gives the matrix

$$R_1^l = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial \psi_2}{\partial \xi_1} & \frac{\partial \psi_2}{\partial \xi_2} & 0 \\ * & * & \frac{\partial \psi_2}{\partial \xi_1}(b_1 - b_3\xi_1) + \frac{\partial \psi_2}{\partial \xi_2}(b_2 - b_3\xi_2) \end{pmatrix} \tag{13}$$

The condition $\text{Rank}\, R_1^l = \text{Rank}\, R_1^r$ yields a PDE

$$\frac{\partial \psi_2}{\partial \xi_1}(b_1 - b_3\xi_1) + \frac{\partial \psi_2}{\partial \xi_2}(b_2 - b_3\xi_2) = 0 \tag{14}$$

whose general solution is

$$\psi_2(\xi_1, \xi_2) = F\left(\frac{b_2 - b_3\xi_2}{b_3(b_1 - b_3\xi_1)}\right) \tag{15}$$

We choose F as the identity function:

$$\psi_2(\xi_1, \xi_2) = \frac{b_2 - b_3\xi_2}{b_3(b_1 - b_3\xi_1)} \tag{16}$$

Next, we solve the vector fields r_i in (10) and obtain a non-unique solution. We express the solutions as

$$r_1 = \begin{pmatrix} 0 & 0 & \frac{1}{b_1 - b_3\xi_1} \end{pmatrix}, \quad r_2 = (0 \;\; b_3\xi_1 - b_1 \;\; \rho) \tag{17}$$

where we assume $\rho = \rho(\xi_1)$ is some function of ξ_1 to be determined. In order to satisfy the Lie bracket conditions (11) we try an output transformation for the first output $\psi_1(\xi_1)$. To satisfy (11) the following differential equations must be satisfied:

$$\frac{d^2\psi_1}{d\xi_1^2}(b_1 - b_3\xi_1) - 2b_3\frac{d\psi_1}{d\xi_1} = 0 \tag{18}$$

$$(b_1 - b_3\xi_1)\left(\frac{d\rho}{d\xi_1}(b_3\xi_1 - b_1) + b_3\omega_3 + \omega_1 b_1 - \rho(\xi_1)b_3\right) = 0 \tag{19}$$

Solving (18) results in

$$\psi_1(\xi_1) = C_1 + \frac{C_2}{b_3\xi_1 - b_1} \tag{20}$$

where we choose $C_1 = 0$ and $C_2 = 1$. Hence,

$$\bar{y}_1 = \frac{1}{b_3\xi_1 - b_1}, \quad \bar{y}_2 = \frac{b_3\xi_2 - b_2}{b_3(-b_1 + b_3\xi_1)} \tag{21}$$

Solving (19) gives

$$\rho(\xi_1) = (b_3\xi_1 - b_1)C_3 + \frac{b_3\omega_3 + \omega_1 b_1}{b_3} \tag{22}$$

and choosing $C_3 = 0$ gives

$$\rho = \frac{b_3\omega_3 + \omega_1 b_1}{b_3} \tag{23}$$

A state transformation $z = \Phi(\xi) = (\Phi_1(\xi), \Phi_2(\xi), \Phi_3(\xi))^T$ can be computed as

$$\begin{aligned} \Phi_1(\xi) =& \frac{1}{2b_3(b_3\xi_1 - b_1)^2}(2\xi_3\xi_1 b_3^3 + (\omega_2 - 2\xi_3 b_1 - 2\omega_3\xi_2)b_3^2 \\ &+ ((2\omega_2\xi_1 - 2\omega_1\xi_2)b_1 + \omega_3 b_2)b_3 - \omega_2 b_1^2 + b_2 b_1 \omega_1) \\ \Phi_2(\xi_i) =& \frac{1}{b_3\xi_1 - b_1} \\ \Phi_3(\xi) =& \frac{b_2 - \xi_2 b_3}{(b_1 - b_3\xi_1)b_3} \end{aligned} \tag{24}$$

Applying the state transformation (24) and the output transformation (21) gives the OF

$$\begin{aligned} \dot{z}_1 &= \eta_1(\bar{y}_1, \bar{y}_2) \\ \dot{z}_2 &= z_1 + \eta_2(\bar{y}_1, \bar{y}_2) \\ \dot{z}_3 &= \eta_3(\bar{y}_1, \bar{y}_2) \\ \bar{y}_1 &= z_2, \; \bar{y}_2 = z_3 \end{aligned} \tag{25}$$

where the functions $\eta_i(\bar{y}_1, \bar{y}_2)$, $i = 1, 2, 3$ are

$$\begin{aligned} \eta_1 =& -\frac{1}{b_3^2}(\bar{y}_1^3 b_3^2 b_2^2 \omega_3^2 + 2\bar{y}_1^3 \omega_2^2 b_1^2 b_3^2 + \bar{y}_1^3 b_2^2 b_1^2 \omega_1^2 \\ &+ \bar{y}_1^3 b_3^4 \omega_2^2 + \bar{y}_1^3 \omega_2^2 b_1^4 - 2\bar{y}_1^3 b_3^3 b_2 \omega_3 \omega_2 - 2\bar{y}_1^3 b_2 b_1^3 \omega_1 \omega_2 \\ &- 2\bar{y}_1^3 b_2 b_1 \omega_1 b_3^2 \omega_2 - 2\bar{y}_1^3 b_3 b_2 \omega_3 \omega_2 b_1^2 + 2\bar{y}_1^3 b_3 b_2^2 \omega_3 b_1 \omega_1 \\ &- 2\bar{y}_1^2 b_3 b_2 \omega_3 \omega_2 b_1 - 3\bar{y}_1^2 b_3^3 \omega_2 \omega_1 b_1 \bar{y}_2 - 3\bar{y}_1^2 \omega_2 b_1^2 b_3^2 \omega_3 \bar{y}_2 \\ &- 3\bar{y}_1^2 b_3^4 \omega_2 \omega_3 \bar{y}_2 - b_3^2 \omega_3 \bar{y}_2 \omega_2 + \omega_1^2 b_1 \bar{y}_2^2 b_3^2 + b_3^3 \omega_3 \bar{y}_2^2 \omega_1 \\ &- 2\omega_1 b_1 \bar{y}_2 b_3 \omega_2 - 5\bar{y}_1 \omega_2 b_1^2 \omega_1 \bar{y}_2 b_3 + 2\bar{y}_1 \omega_1^2 b_1^2 \bar{y}_2^2 b_3^2 \\ &- 4\bar{y}_1 b_3^2 \omega_3 \bar{y}_2 \omega_2 b_1 + 4\bar{y}_1 \omega_1 b_1 \bar{y}_2^2 b_3^3 \omega_3 + \bar{y}_1 b_3 \omega_1 b_1 \omega_3 \\ &+ \bar{y}_1 b_2 b_1 \omega_1^2 \bar{y}_2 b_3 - \bar{y}_1 \omega_1 b_2 b_1 \omega_2 + \bar{y}_1 b_3^2 b_2 \omega_3 \omega_1 \bar{y}_2 \\ &+ 3\bar{y}_1^2 \omega_2^2 b_1^3 + 3\bar{y}_1 \omega_2^2 b_1^2 + \bar{y}_1 b_3^2 \omega_3^2 + \bar{y}_1^2 b_3^2 \omega_3^2 b_1 \\ &- \bar{y}_1^2 b_3^3 \omega_3 \omega_1 - \bar{y}_1^2 \omega_1^2 b_1 b_3^2 + 3\bar{y}_1^2 b_3^2 \omega_2^2 b_1 + 2\bar{y}_1 b_3^4 \omega_3^2 \bar{y}_2^2 \\ &- \bar{y}_1 b_3^3 \omega_2 \omega_1 \bar{y}_2 - 3\bar{y}_1^2 \omega_2 b_1^3 \omega_1 \bar{y}_2 b_3 - \bar{y}_1^2 b_3^2 \omega_2 \omega_1 b_2 \\ &+ \bar{y}_1^2 \omega_1 b_1^2 b_3 \omega_3 + 3\bar{y}_1^2 b_3^3 b_2 \omega_3^2 \bar{y}_2 - 3\bar{y}_1^2 b_2 b_1^2 \omega_1 \omega_2 \\ &+ 3\bar{y}_1^2 b_2 b_1^2 \omega_1^2 \bar{y}_2 b_3 + 6\bar{y}_1^2 b_3^2 b_2 \omega_3 \omega_1 b_1 \bar{y}_2 + \bar{y}_1 b_3^2 \omega_2^2 + \omega_2^2 b_1) \\ \eta_2 =& \frac{1}{2b_3}(-2\omega_2 + 3b_3 b_2 \omega_3 \bar{y}_1^2 + 3b_2 b_1 \omega_1 \bar{y}_1^2 - 3b_3^2 \omega_2 \bar{y}_1^2 \\ &- 3\omega_2 b_1^2 \bar{y}_1^2 - 6\omega_2 b_1 \bar{y}_1 + 2\omega_1 \bar{y}_2 b_3 + 2\omega_1 b_2 \bar{y}_1 \\ &+ 4\bar{y}_1 \omega_1 b_1 \bar{y}_2 b_3 + 4\bar{y}_1 b_3^2 \omega_3 \bar{y}_2) \\ \eta_3 =& \frac{1}{b_3^2}(-\bar{y}_1 \omega_1 b_2^2 - \bar{y}_1 \omega_1 b_3^2 + \bar{y}_1 b_3 \omega_3 b_1 - \bar{y}_1 b_3^3 \omega_2 \bar{y}_2 \\ &+ \bar{y}_1 b_2 \omega_2 b_1 + \bar{y}_1 b_2 \omega_1 b_1 \bar{y}_2 b_3 + \bar{y}_1 b_2 b_3^2 \omega_3 \bar{y}_2 \\ &- \bar{y}_1 \omega_2 b_1^2 \bar{y}_2 b_3 + b_3 \omega_3 + b_2 \omega_2 + \omega_1 b_1 \bar{y}_2^2 b_3^2 + b_3^3 \omega_3 \bar{y}_2^2 \\ &- \omega_2 \bar{y}_2 b_3 b_1 - \omega_1 \bar{y}_2 b_3 b_2) \end{aligned}$$

The above derivation of the OF illustrates a procedure where the output transformation is solved so that rank conditions (9) and Lie bracket conditions (11) are satisfied. At the same time, the procedure utilizes a degree of freedom in determining r_i, $i = 1, 2$ from (10). We remark that if observability indices are equal, then no degree of freedom results from (10).

The state transformation (24) and the output transformation (21) resulting in the OF (25) require $b_3 \neq 0$. The case $b_3 = 0$ can also be handled using the same procedure, however instead using the linear output transformation

$$\bar{y}_1 = \xi_1, \quad \bar{y}_2 = b_1\xi_2 - b_2\xi_1 \tag{26}$$

which is valid when $b_1 \neq 0$. For the case $b_1 = b_3 = 0$, an output transformation is not required, as shown in Dahl et al. [2007a].

The above approach can also be applied to the planar perspective system

$$\dot{x} = Ax + b, \quad y = \frac{x_1}{x_2}$$

with

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

which does not admit an OF without output transformation Soatto and Perona [1994]. The details of the procedure are straightforward and not provided.

3.2 Characteristic Equation Approach

The method described in subsection 3.1 utilizes conditions given in Xia and Gao [1989] and Marino and Tomei [1995] to compute the output transformation. Alternatively, one can use a method based on a so-called Generalized Characteristic Equation (GCE) Keller [1987]. For a two output system with observability indices $(2,1)$, the GCEs are

$$\begin{aligned} L_f^2\psi_1(y) =& L_f\gamma_2(\psi(y)) + \gamma_1(\psi(y)) \\ L_f\psi_2(y) =& \gamma_3(\psi(y)) \end{aligned}$$

where $\psi = (\psi_1, \psi_2)^T$. Expanding the GCEs and performing coefficient matching leads to necessary and sufficient conditions on the transformability to OF. In particular the so-called polynomial condition results: $\partial^2 L_f^2\psi_1(y)/\partial\dot{y}_1^2 = 0$ and $\partial L_f\psi_2(y)/\partial\dot{y}_1 = 0$. We assume the output transformation for the first subsystem to only depend on y_1, i.e. $\bar{y}_1 = \psi_1(y_1)$. We are able to solve for ψ_1 s.t. the system satisfies a polynomial condition. That is, $L_f^2\bar{y}_1$ is linear in $\dot{y}_1$ with coefficients depending on y:

$$\begin{aligned} \dot{\bar{y}}_1 =& \frac{d\psi_1}{dy_1}\dot{y}_1 \\ \ddot{\bar{y}}_1 =& L_f\left(\frac{d\psi_1}{dy_1}\right)\dot{y}_1 + \frac{d\psi_1}{dy_1}L_f\dot{y}_1 \\ =& \frac{d^2\psi_1}{dy_1^2}\dot{y}_1^2 + \frac{d\psi_1}{dy_1}\underbrace{\left(\alpha_2(y)\dot{y}_1^2 + \alpha_3(y)\dot{y}_1 + \alpha_4(y)\right)}_{L_f^2 h_1} \end{aligned} \tag{27}$$

In order to remove the dependence on $\dot{y}_1^2$ on the RHS of (27) we have the ordinary differential equation (ODE):

$$\frac{d^2\psi_1}{dy_1^2} + \alpha_2(y)\frac{d\psi_1}{dy_1} = 0 \tag{28}$$

where

$$\alpha_2(y) = \frac{2b_3}{b_3y_1 - b_1} \tag{29}$$

We notice that the ODE (28) with $\alpha_2(y)$ given by (29) is the same as (18), hence the output transformation for the first subsystem is given by (20), where we, as done in subsection 3.1, choose ψ_1 by taking $C_1 = 0$ and $C_2 = 1$.

For the second subsystem we assume a more general dependence for the output transformation: $\psi_2(y)$. Following the similar procedure as that used for the first subsystem we have

$$\dot{\bar{y}}_2 = \frac{\partial\psi_2}{\partial y_1}\dot{y}_1 + \frac{\partial\psi_2}{\partial y_2}\dot{y}_2 = \frac{\partial\psi_2}{\partial y_1}\dot{y}_1 + \frac{\partial\psi_2}{\partial y_2}\underbrace{\left(\alpha_5(y)\dot{y}_1 + \alpha_6(y_1)\right)}_{L_f h_2}$$

and the partial differential equation (PDE)

$$\frac{\partial\psi_2}{\partial y_1} + \frac{\partial\psi_2}{\partial y_2}\alpha_5(y) = 0 \tag{30}$$

where

$$\alpha_5(y) = \frac{b_3y_2 - b_2}{b_3y_1 - b_1} \tag{31}$$

One can see that the PDE (30) with $\alpha_5(y)$ given by (31) is the same as (14). Hence, the output transformation $\psi_2(y)$ for the second subsystem is given by (16).

3.3 Time-scaled block triangular observer form

The system (4) in observable form is already in BTF Wang and Lynch [2006a]. We attempt to transform the first subsystem to BTOF Wang and Lynch [2007]. Defining the observable coordinates as $\zeta = (\zeta_1^T, \zeta_{21})^T = (\zeta_{11}, \zeta_{12}, \zeta_{21})^T = (h_1(\xi), L_f h_1(\xi), h_2(\xi))^T$, one can compute the starting vector $g_1 = \partial/\partial\zeta_{12}$ according to [Wang and Lynch, 2006b, Eq. (6)] and verify that the Lie bracket [Wang and Lynch, 2006b, Eq. (7)] is not satisfied. We introduce the time scaling transformation for the first subsystem

$$\frac{d\tau_1}{dt} = s_1(y) > 0$$

where $s_1(y)$ is the time scaling function (TSF) to be determined. We apply [Wang and Lynch, 2006b, Prop. 3.1]

$$dL_{g_i}L_{F^i}^{\lambda_i}h_i = l_{\lambda_i}\frac{1}{s_i}\frac{\partial s_i}{\partial y_i}dL_{F^i}h_i \mod \{dz_k^j, 1 \le k \le \lambda_j, 1 \le j \le i-1, dz_1^i\}$$

for the $i = 1$ subsystem, with $\lambda_1 = 2$, $F^1 = f_1 = \zeta_{12}\partial/\partial\zeta_{11} + (L_f^2 h_1(\zeta))\partial/\partial\zeta_{12}$, $l_2 = 2$, and $g_1 = \partial/\partial\zeta_{12}$. This yields the PDE for s_1

$$\frac{4b_3}{b_3y_1 - b_1} = \frac{2}{s_1}\frac{\partial s_1}{\partial y_1}$$

Solving this PDE yields the time scaling transformation

$$\frac{d\tau_1}{dt} = (b_3y_1 - b_1)^2 = s_1(y) > 0$$

Defining $\bar{f}_1 = f_1/s_1$ and calculating the vector fields $\bar{g}_1 = s_1 g_1, ad_{-\bar{f}_1}\bar{g}_1$, we can verify the Lie bracket condition $[\bar{g}_1, ad_{-\bar{f}_1}\bar{g}_1] = 0$. However, [Wang and Lynch, 2006b, Eq. (8)] requires

$$\frac{\partial}{\partial y_2}ad_{-f_1}\bar{g}_1 = 0$$

which is satisfied if and only if

$$\omega_1 b_1 + \omega_3 b_3 = 0. \tag{32}$$

This constraint also appears in the dynamic error linearization in Dahl et al. [2007a]. Given (32), the transformation of state can be solved from

$$\frac{\partial\Phi_1(\zeta_1)}{\partial\zeta_1}[ad_{-\bar{f}_1}\bar{g}_1, \bar{g}_1] = \mathbf{I}_2$$

where

$$ad_{-\bar{f}_1}\bar{g}_1 = \begin{pmatrix} 1 \\ \dfrac{3\omega_2\zeta_{11}b_1 - 2b_3\zeta_{12} + 3b_3\omega_2 - \omega_3 b_2 - \omega_1\zeta_{11}b_2}{b_1 - b_3\zeta_{11}} \end{pmatrix}$$

This gives the transformation $z = \Phi(\zeta) = (\Phi_1, \Phi_2, \Phi_3)^T$ to TBTOF:

$$\begin{aligned} \Phi_1 =& \zeta_{11} \\ \Phi_2 =& \frac{1}{2b_3^2(b_1 - b_3\zeta_{11})^2}(2\zeta_{12}b_3^2 - \omega_1 b_1 b_2 - 3b_3^2\omega_2 \\ &+ 3\omega_2 b_1^2 + \omega_3 b_2 b_3 + 2\omega_1 b_2 b_3\zeta_{11} - 6\omega_2 b_1 b_3\zeta_{11}) \\ \Phi_3 =& \zeta_{21} \end{aligned}$$

where we have reused the notation for z and Φ. Applying $\Phi(\zeta)$ to the system in observable form gives

$$\begin{pmatrix} \frac{dz_{11}}{d\tau_1} \\ \frac{dz_{12}}{d\tau_1} \\ \frac{dz_{21}}{dt} \end{pmatrix} = \begin{pmatrix} z_{12} + \beta_{11}(z_{11}, y_2) \\ \beta_{12}(z_{11}, y_2) \\ \beta_{21}(z_1, z_{21}) \end{pmatrix}$$

The TBTOF allows for a straightforward observer design

$$\begin{pmatrix} \frac{d\hat{z}_1}{d\tau_1} \\ \frac{d\hat{z}_{21}}{dt} \end{pmatrix} = \begin{pmatrix} A_1\hat{z}_1 + \beta_1 + L_1C_1(z_1 - \hat{z}_1) \\ \hat{\beta}_{21} + L_2C_2(z_{21} - \hat{z}_{21}) \end{pmatrix}$$

where $\hat{z}_1 = (\hat{z}_{11}, \hat{z}_{12})^T, A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, C_1 = [1, 0]^T, \beta_1 = (\beta_{11}, \beta_{12})^T, C_2 = 1, \hat{\beta}_{21} = \hat{\beta}_{21}(\hat{z}_{12}, y)$, and L_1, L_2 are chosen so that $A_i - L_iC_i$ is Hurwitz. The corresponding error dynamics in the new time scale is

$$\begin{pmatrix} \frac{d\tilde{z}_1}{d\tau_1} \\ \frac{d\tilde{z}_{21}}{dt} \end{pmatrix} = \begin{pmatrix} A_1 - L_1C_1 & 0 \\ \mathbf{0} & -L_2C_2 \end{pmatrix} \tilde{z} + \begin{pmatrix} \mathbf{0} \\ \beta_{21} - \hat{\beta}_{21} \end{pmatrix}$$

whose zero solution is globally exponentially stable (GES). Assuming there exist positive constants T_0, ε such that

$$\int_t^{t+T_0} s_1(\xi)\mathrm{d}\xi \geq \varepsilon, \quad \forall t \geq t_0$$

we conclude the zero solution of the error dynamics is GES in the original time. The observer in x-coordinates and t time is

$$\dot{\hat{\zeta}} = \begin{pmatrix} \frac{s_1(y)}{s_1(\hat{y})} f_1(\hat{x}) \\ f_2(\hat{x}) \end{pmatrix} + \left(\frac{\partial \hat{z}}{\partial \hat{x}}\right)^{-1} \begin{pmatrix} s_1(y)\left(\beta_1 - \hat{\beta}_1^* + L_1(y_1 - C_1\hat{z}_1)\right) \\ \hat{\beta}_{21} - \hat{\beta}_{21}^* + L_2(y_2 - C_2\hat{z}_{21}) \end{pmatrix}$$

where $\hat{\beta}_1^* = \hat{\beta}_1(\hat{z}_{11}, \hat{y}_2), \hat{\beta}_{21}^* = \hat{\beta}_{21}(\hat{z}_1, \hat{y}_2)$.

4. SIMULATIONS

We simulate the observers with motion parameters $\omega = (-1, 1, 1)^T, b = (1, 2, 1)^T$, the observer gain chosen to place the eigenvalues of error dynamics at -4, and the initial conditions (ICs) in the format of $(x_1, x_2, x_3, \hat{x}_1, \hat{x}_2, \hat{x}_3)^T$:

$$\begin{aligned} IC1 &: (-1, 2, 2, -1/6, 1/3, 1/3)^T \\ IC2 &: (-1, 2, 1, -0.03, 0.12, 0.30)^T \\ IC3 &: (-2, 3, 4, -0.4, 2.4, 0.4)^T \end{aligned} \quad (33)$$

We perform observer design based on OF and TBTOF. For the OF-based observer, plots of the norm of the error in the observer coordinates $\|\tilde{z}\| = \|z - \hat{z}\|$ and in the original coordinates $\|\tilde{x}\| = \|x - \hat{x}\|$ are presented in Figures 1 and 2, using the colors red, green, and blue for the three initial conditions in (33). For the TBTOF-based observer, the corresponding simulation results are given in Figures 3 and 4.

5. CONCLUSIONS

This paper has shown that a perspective system admits two observer forms. These observer forms naturally lead to observer designs with error dynamics which are easy to stabilize. The first observer form is the OF with output transformation which provides error convergence without motion constraints (assuming constant motion parameters). The second observer form is a TBTOF which requires the same motion constraint as in previous work Dahl et al. [2007a] on dynamic error linearization. Future work involves generalizing the normal form-based approach to

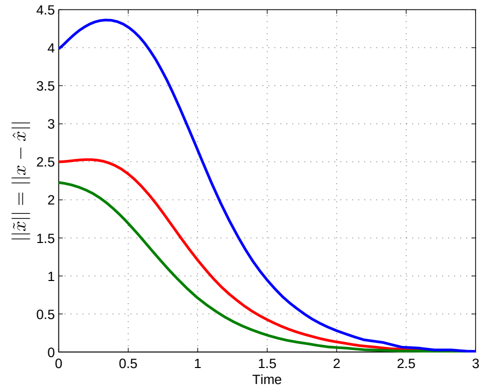

Fig. 1. Norm of state estimate error in x-coordinates using an observer form with output transformation.

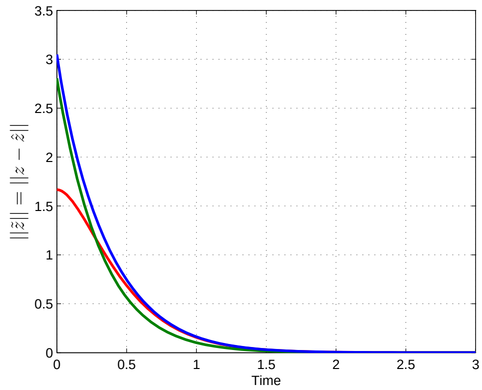

Fig. 2. Norm of state estimate error in z-coordinates using an observer form with output transformation.

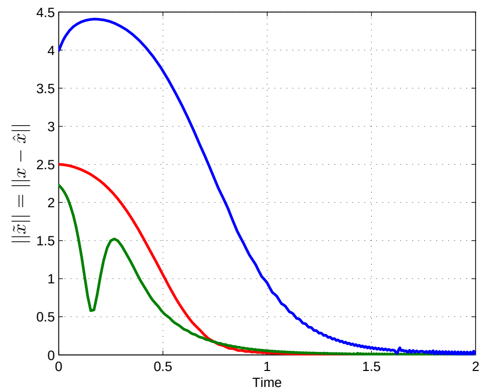

Fig. 3. Norm of state estimate error in x-coordinates using a TBTOF observer.

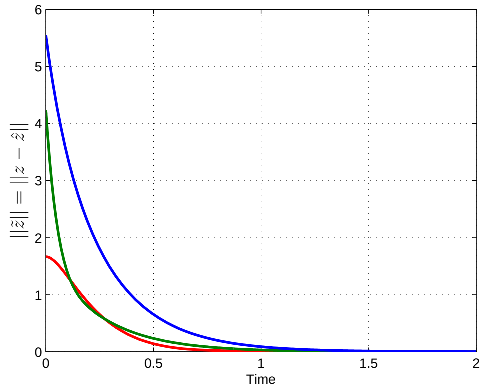

Fig. 4. Norm of state estimate error in z-coordinates using a TBTOF observer.

allow for time-varying and/or unknown motion parameters.

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