Exact Observer Error Linearization for Perspective Dynamic Systems

O. Dahl and A. Heyden
Applied Mathematics Group, Malmo University, Malmo, Sweden
{ola.dahlanders.heyden}@mah.se

Abstract

Estimation of 3D position information from 2D images in computer vision systems can be formulated as a state estimation problem for a nonlinear perspective dynamic system. The state estimation can be performed using different kinds of nonlinear observers. In this paper we investigate observer error linearization, where the goal is to find a coordinate transformation that results in a system for which a linear observer can be constructed. It is shown that using a state transformation combined with an output transformation, the system admits an observer form which leads to an observer with linear error dynamics.

1 Introduction

The problem of estimating 3D structure and motion from 2D perspective observations can be formulated using a nonlinear perspective dynamic system. The perspective system is obtained by considering the relative motion between a perspective camera and an observed object. The estimation of both structure and motion can be achieved by an observer for states and parameters. Existing approaches have used the extended Kalman filter [2, 17] or adaptive observers [4, 7]. The problem of estimating structure when the motion parameters are measured or otherwise assumed available, has been considered using observer-based approaches in [16, 11, 3, 9, 8, 1, 10, 15].

This paper presents structure estimation results, showing how a perspective system can be transformed into an observer form. These forms naturally lead to observers with simple error dynamics systems. The simplicity of the error dynamics leads to a straightforward stability analysis. Relative to existing related work, the results here show that it is possible to achieve linear time-invariant error dynamics without any constraints on the type of motion which potentially limited the application of the approach.

2 Background

2.1 Perspective dynamic systems

A perspective dynamic system with three states and two outputs, derived assuming a calibrated pinhole camera and observations of feature points on a rigid object, can be written as e.g. [1], [13]:

\[
\dot{x} = Ax + b, \quad y = \left( \frac{\omega_1}{x_3} \frac{\omega_2}{x_3} \right)^T \tag{1}
\]

with

\[
A = \begin{pmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{2}
\]

where we assume \(\omega_i, b_i, 1 \leq i \leq 3\) are constant.

As in e.g. [3], a useful alternative formulation of the perspective dynamic system (1) can be obtained by applying an initial change of coordinates

\[
\xi = \left( \xi_1 \quad \xi_2 \quad \xi_3 \right)^T = \left( \frac{\omega_1}{x_3} \frac{\omega_2}{x_3} \frac{1}{x_3} \right)^T \tag{3}
\]

which results in

\[
\dot{\xi}_1 = -\omega_1 \xi_1 \xi_2 + \omega_2 (1 + \xi_3) - \omega_3 \xi_2 + (b_1 - b_3 \xi_1) \xi_3 \\
\dot{\xi}_2 = \omega_2 \xi_1 \xi_2 - \omega_1 (1 + \xi_3) + \omega_3 \xi_1 + (b_2 - b_3 \xi_2) \xi_3 \\
\dot{\xi}_3 = -(\omega_1 \xi_2 - \omega_2 \xi_1 + b_3 \xi_3) \xi_3 \\
y_1 = \xi_1, \quad y_2 = \xi_2 \tag{4}
\]

where the nonlinear terms now occur in the state equations, and the output equations are linear.

2.2 Observability

We use the notation \(L_f h(x)\) for the Lie derivative of a function \(h(x)\) along a vector field \(f(x)\) and the notation \(L_f^k h(x)\) for the \(k\) times repeated Lie derivative, together with the notation \(d\lambda(x)\) for the gradient of a function \(\lambda(x)\). Given two vector fields \(f(x)\) and
Given the dynamic system (5), the existence conditions for a change of state coordinates under which the system (5) admits an OF are well-established, [12, 14, 19]. System (4) is transformable to OF by a state transformation \( \Phi(\xi) \) if and only if the following three conditions are fulfilled:

1. The matrices \( R_j^1 \) and \( R^j_2 \)

\[
R^1_j = \{ \text{all } \partial L^{k}_{i} h_{i} : 0 \leq k \leq k_j - 1, i \neq j, 1 \leq i \leq 2, dL^{k}_{i} h_{j} : 0 \leq k \leq k_j - 2 \} \\
R^2_j = \{ \text{all } \partial L^{k}_{i} h_{i} : 0 \leq k \leq \min(k_i, k_j) - 1, i \neq j, 1 \leq i \leq 2, dL^{k}_{i} h_{j} : 0 \leq k \leq k_j - 2 \}
\]

2. There exist vector fields \( r_i, 1 \leq i \leq 2 \), such that

\[
L_{r_i} \partial L^{k-1}_{i} h_j = \delta_{i,j} \delta_{k_j,1}, \\
1 \leq i \leq 2, \quad 1 \leq k \leq k_i, \quad 1 \leq j \leq 2
\]

where \( \delta_{i,j} = 1 \) when \( i = j \) and zero otherwise.

3. \([ad^{k-1}_{b_j} r_i, ad^{k-1}_{b_j} r_j] = 0, 1 \leq i,j \leq 2, \quad 0 \leq k \leq k_i - 1, \quad 0 \leq l \leq k_j - 1\]

Without output transformation, the system (4) admits an OF under the constraint \( b_2 = b_3 = 0 \) [6] given the observability assumption \( b_1 - b_3 \xi_1 \neq 0 \). In this paper we provide a result which extend the work in [6]. This result shows the existence of an output transformation \( \bar{y} = \Psi(y) \) and a state transformation \( z = \Phi(\xi) \) such that (4) is transformable to OF without motion constraints.

### 3 Observer forms for perspective systems

This section presents our main results regarding the transformation of (4) to observer form. We follow an approach to derive the output transformation \( \bar{y} = \Psi(y) \), and give the state transformation \( z = \Phi(\xi) \) to observer form. The results have been derived by using a Maple library for observer error linearization [5].

#### 3.1 Observer form

An OF for the perspective system in \( \xi \)-coordinates (4) can be derived by first finding an output transformation, and then computing a state transformation. An initial observation is that the rank condition (9) is in general not satisfied. This can be seen from the matrices \( R^1_j, R^2_j \) for the first output, i.e. \( j = 1 \),

\[
R^1_j = \begin{pmatrix}
\partial L^{k}_{1} h_{1} \\
\partial L^{k}_{1} h_{2}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

which have different rank unless \( b_2 - b_3 \xi_2 \neq 0 \). Given that no output transformation is employed, the rank condition can be satisfied when \( b_2 = b_3 = 0 \), a condition which is used in [6] to derive an observer form for the perspective system (4). The ranks of \( R^1_j, R^2_j \) can be made equal if an output transformation

\[
\bar{y}_1 = \xi_1, \quad \bar{y}_2 = \psi_2(\xi_1, \xi_2)
\]
is used. This gives the matrix
\[
\begin{pmatrix}
\frac{∂ψ_3}{∂ξ_1} & \frac{∂ψ_3}{∂ξ_2} & 0 \\
\frac{∂ψ_2}{∂ξ_1} & \frac{∂ψ_2}{∂ξ_2} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
\[
R'_1 = \left( \begin{array}{ccc}
\frac{∂ψ_3}{∂ξ_1} & \frac{∂ψ_3}{∂ξ_2} & 0 \\
\frac{∂ψ_2}{∂ξ_1} & \frac{∂ψ_2}{∂ξ_2} & 0 \\
0 & 0 & 0
\end{array} \right)
\]

The condition \( \text{Rank } R'_1 = \text{Rank } R'_1 \); yields a PDE
\[
\frac{∂ψ_2}{∂ξ_1}(b_1 - b_3ξ_1) + \frac{∂ψ_2}{∂ξ_2}(b_2 - b_3ξ_2) = 0
\]
\[
(14)
\]

whose general solution is
\[
ψ_2(ξ_1, ξ_2) = F(\frac{b_2 - b_3ξ_2}{b_3(1 - b_3ξ_1)})
\]
\[
(15)
\]
where we choose \( F \) as the identity function.

Next, we solve the vector fields \( r_i \) in (10) and obtain a non-unique solution expressed as
\[
r_1 = \left( 0, 0, \frac{1}{b_3ξ_1 - b_1} \right), \quad r_2 = \left( 0, b_3ξ_1 - b_1, ρ \right)
\]
\[
(16)
\]
where we assume \( ρ = ρ(ξ_1) \) is some function of \( ξ_1 \) to be determined. In order to satisfy the Lie bracket conditions (11) we try an output transformation for the first output \( ψ_i(ξ_1) \). To satisfy (11) the following differential equations must be satisfied:
\[
\frac{d^2ψ_1}{dξ_1^2}(b_1 - b_3ξ_1) - 2b_3 \frac{dψ_1}{dξ_1} = 0
\]
\[
(17)
\]
\[
(18)
\]
Solving (17) results in
\[
ψ_1(ξ_1) = C_1 + \frac{C_2}{b_3ξ_1 - b_1}
\]
\[
(19)
\]
where we choose \( C_1 = 0 \) and \( C_2 = 1 \). Hence,
\[
\bar{y}_1 = \frac{1}{b_3ξ_1 - b_1}, \quad \bar{y}_2 = \frac{b_3ξ_2}{b_3(-b_1 + b_3ξ_1)}
\]
\[
(20)
\]
Solving (18) gives
\[
ρ(ξ_1) = (b_3ξ_1 - b_1)C_3 + \frac{b_3ω_3 + ω_1b_1}{b_3}
\]
\[
(21)
\]
and we choose \( C_3 = 0 \). A state transformation \( z = \Phi(ξ) \) can be computed as
\[
\Phi_1(ξ) = \frac{1}{2b_3(b_3ξ_1 - b_1)}(2ξ_2b_5b_3^2 + (ω_2 - 2ξ_1b_1 - 2ω_3ξ_2)b_3^2
\]
\[
+ (2ω_2ξ_1 - 2ω_1ξ_1)b_1 + ω_3b_2)b_3 - ω_2b_3^2 + b_2b_1ω_1)
\]
\[
\Phi_2(ξ) = \frac{1}{b_3ξ_1 - b_1}
\]
\[
\Phi_3(ξ) = \frac{b_3ξ_2}{(b_3ξ_1 - b_1)b_3}
\]
\[
(22)
\]
Applying the state transformation (22) and the output transformation (20) gives the OF
\[
\bar{z}_1 = η_1(\bar{y}_1, \bar{y}_2)
\]
\[
\bar{z}_2 = \bar{z}_1 + η_2(\bar{y}_1, \bar{y}_2)
\]
\[
\bar{z}_3 = η_3(\bar{y}_1, \bar{y}_2)
\]
\[
\bar{y}_1 = \bar{z}_2, \quad \bar{y}_2 = \bar{z}_3
\]
\[
(23)
\]
where the functions \( η_i(\bar{y}_1, \bar{y}_2), i = 1, 2, 3 \) are rather complicated polynomial functions. The above derivation of the OF illustrates a procedure where the output transformation is solved so that rank conditions (9) and Lie bracket conditions (11) are satisfied. At the same time, the procedure utilizes a degree of freedom in determining \( r_i, i = 1, 2 \) from (10). We remark that if observability indices are equal, then no degree of freedom results from (10).

The state transformation (22) and the output transformation (20) resulting in the OF (23) require \( b_3 \neq 0 \). The case \( b_3 = 0 \) can be handled similarly. For the case \( b_1 = b_3 = 0, b_2 \neq 0 \) and the case \( b_2 = b_3 = 0, b_1 \neq 0 \) an output transformation is not required, as shown in [6]. Also note that the case \( b_1 = b_2 = 0 \) is not observable.

The above approach can also be applied to the planar perspective system
\[
\dot{x} = Ax + b, \quad y = \frac{x_1}{x_2}
\]
\[
(24)
\]
with
\[
A = \begin{pmatrix} 0 & -ω \\ ω & 0 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}
\]
\[
(25)
\]
which does not admit an OF without output transformation [18]. The details of the procedure are straightforward and not provided.

4 Simulations

We simulate the observer with motion parameters \( ω = (-1, 1, 1)^T \), \( b = (1, 2, 1)^T \), the observer gain chosen to place the eigenvalues of error dynamics at \(-4 \) (observe that the poles of the error dynamics can be chosen freely and that stability is guaranteed), and the initial conditions (ICs) in the format of \( (x_1, x_2, x_3, x_1, x_2, x_3)^T \):
\[
IC1 : (-1, 2, 2, -1/6, 1/3, 1/3)^T
\]
\[
IC2 : (-1, 2, 1, 0.03, 0.12, 0.30)^T
\]
\[
IC3 : (-2, 3, 4, -0.4, 2.4, 0.4)^T
\]
We perform observer design based on OF. Plots of the norm of the error in the observer coordinates \( \| z - \hat{z} \| \) and in the original coordinates \( \| x - \hat{x} \| \) are presented in Figures 1 and 2, using the colors red, green, and blue for the three initial conditions in (24).
5 Conclusions

This paper has shown that a perspective system admits an observer forms. This observer form naturally lead to observer designs with error dynamics which are easy to stabilize. The observer form is the OF with output transformation which provides error convergence without motion constraints (assumming constant motion parameters). Future work involves generalizing the normal form-based approach to allow for time-varying and/or unknown motion parameters.

References