Abstract

The watchman route problem is a well investigated problem in which a closed tour inside a polygon that satisfies visibility constraints is optimized. We explore a new variant in which a tour needs to see a set of locations inside a polygon so that the maximum time interval in which those locations are not guarded (the delay) is minimized. We call such tours surveillance tours. We show that an algorithm that follows the ideas of the classic watchman problem provides a 2-approximation to our problem. We then investigate the case where each location is associated with a weight that represents the importance of not leaving the location unguarded for long periods. Thus, the tour is required to see locations with larger weights more frequently. We show that this variant is NP-hard and present two approximation algorithms.

1 Introduction

Visibility coverage of polygons with guards (mainly known as Art Gallery problems) have been central geometric problems for many years. Usually guards are defined as static points that see in any direction for any distance and visibility is defined by the clearance of straight lines between two features (in other words, two features see each other if the segment that connects them does not intersect (the interior of) any other feature of the input). Coverage is achieved if any point inside the polygon is visible by at least one guard. Several art gallery theorems have been proposed for different kind of settings [3].

Allowing a guard to move inside the polygons defines a related problem but yet with very different properties. Here, a set of mobile guards walk on closed cycles (also called tours or routes) so that any point inside the polygon is seen by at least one guard during its walk along the tour. The number of guards is a parameter of the problem and the measure criteria relates to the length of the tours (e.g., minimize the longest tour). Several solutions have been proposed for the case of a single mobile guard, a shortest watchman route in a simple polygon. The currently fastest one combines algorithms by Tan [4] and Dror et al. [1], to achieve asymptotic running time $O(n^3 \log n)$.

A variant of the shortest watchman route problem is where the covering is restricted to a given finite subset of points inside the polygon instead of the entire polygon. For simple polygons, this variant is solvable in a similar way as the shortest watchman route problem; see Section 2.

We want to guard a given simple polygon $P$, but rather than finding a shortest tour that covers the points of $P$, we are interested in a tour that minimizes the maximum duration in which any of the points in $P$ are not guarded. We call such a tour a minimum surveillance route for the polygon, abbreviated MSR. Kamphans and Langetepe [2] study a similar concept (inspection paths) but their optimization measure is the sum of the durations where features are not covered rather than the maximum duration.

We show that the two objective functions, minimize the length of the tour and minimize the maximum duration in which any point in the polygon is not guarded, have different optimal tours and that a solution to the shortest watchman route problem is a 2-approximation to the minimum surveillance route problem. We also consider a discrete version of the minimum surveillance tour problem where a given finite subset $S$ of points in the polygon is to be guarded (DMSR). We further generalize this version of the problem by associating priorities to the points of $S$, abbreviated WDMSR. Note that WDMSR may be interesting in situations when not all points are equally important; some need to be guarded more frequently than others. We formulate this idea and show that solving it even in simple polygons is NP-hard. We then propose two approximation algorithms.

2 Preliminaries

The different solutions proposed for the shortest watchman route problem in a simple polygon $P$ identify a subset of the convex vertices of $P$ and computes the shortest tour that visits the visibility polygons of these vertices [1, 4]. (The boundary of these visibility polygons are segments interior to $P$, collinear to boundary edges, called essential cuts, that the shortest watchman route needs to intersect.) Indeed, the algorithms work completely independently from how one defines the points to be guarded. When we just want to guard a finite set of points $S \subset P$, we can exchange the visibility polygons of the convex polygon vertices for the visibility polygons for the points in...
The counterexample in Figure 1 (due to Langetepe) shows that a shortest watchman route and a minimum surveillance route are not necessarily the same. With appropriate scaling, the length of the shortest watchman route, the dotted green triangle in Figure 1(a), has length $3\sqrt{3} \approx 5.19615$ and the delay is the same value. The tour in Figure 1(b), the dotted blue hexagon, has length 6 but only delay 5, since every point in the polygon sees at least one unit length except these endpoints intersects $\mathcal{V}(v)$. Thus, $d(T) \geq ||X||$ and following $X$ from $p'$ to $p$ and back forms a watchman route. Hence,

\[
d(W) \leq ||W|| \leq 2||X|| \leq 2d(T).
\]

A simple modification of the above proof allows us to extend the relationship between the discrete variants of the shortest watchman route and minimum surveillance route problems in a simple polygon $\mathcal{P}$, given a finite set $\mathcal{S}$ of points to guard.

### 4 Weighted Discrete Surveillance Routes

To associate weights (or priorities) to the given points of $\mathcal{S}$, we modify DMSR as follows. Let $\mathcal{P}$ be a simple polygon and let $\mathcal{S}$ be a finite set of points inside $\mathcal{P}$. To each point $p \in \mathcal{S}$ is associated a weight $w(p)$. The idea is that points with higher weights have higher priority and need to be guarded more often than ones with lower weights. Given some tour $T$, we define the weighted delay as

\[
d_w(T) \defeq \max_{p \in \mathcal{S}}\{w(p) \cdot h_{\mathcal{C}}(p)\}.
\]

We are ready to formulate WDMSR.

**Definition 1** WDMSR: Given a simple polygon $\mathcal{P}$, a finite set $\mathcal{S}$ of points inside $\mathcal{P}$, and a weight function $w$ on the points in $\mathcal{S}$, find a tour $T$ such that $d_w(T)$ is minimized.

For simplicity we assume that all weights are positive and that the smallest weight is equal to 1.

#### 4.1 Hardness of WDMSR

The Integer Partition problem is defined as follows.

**Input:** A finite set $\mathcal{Z}$ of positive integers.

**Question:** Is there a subset $\mathcal{Z}' \subseteq \mathcal{Z}$ such that $\sum_{a \in \mathcal{Z}'} a = \sum_{a \in \mathcal{Z} \setminus \mathcal{Z}'} a$?

Since Integer Partition is NP-complete we have the following theorem.

**Theorem 2** WDMSR is NP-hard.

**Proof.** We show a reduction from the Integer Partition problem as illustrated in Figure 2. Given a set $\mathcal{Z}$ of positive integers, we construct a polygon $\mathcal{P}$ consisting of a thin horizontal corridor of width $\epsilon \ll 1$ with $|\mathcal{Z}| + 1$ legs, $L_0, \ldots, L_{|\mathcal{Z}|}$, attached to it. Each leg also has width $\epsilon$ and has a pocket at its end; see Figure 2. Leg $L_i$, $1 \leq i \leq |\mathcal{Z}|$ has length $a_i/2$ from the corridor to the pocket, where $a_i$ is the $i$th integer in $\mathcal{Z}$. Leg $L_0$ has length $1/12 + 2\epsilon$ and is directed upward. We further assume that the values $a_i \in \mathcal{Z}$ are sorted in non-decreasing order.

The points of $\mathcal{S}$ are the convex vertices adjacent to the horizontal edges of each pocket with $p_i \in \mathcal{S}$ in $L_i$. Point $p_0$ has weight $w(p_0) = 2$, marked blue in Figure 2 and the remaining points have weight $w(p_i) = 1$, for $1 \leq i \leq |\mathcal{Z}|$. The visibility polygons $\mathcal{V}(p_i)$ of the points in $\mathcal{S}$ are the regions marked red and blue in Figure 2.
We compress this structure horizontally so that its width is at most $1/6 + 2\epsilon$ and let $\epsilon \leq 1/(18|Z|)$. The polygon consists of $5(|Z| + 1)$ vertices.

Let $Z = \sum_{a \in Z} a$ and we next prove that $Z$ can be partitioned into two equal sets if and only if there is a tour $T$ in $P$ that sees the points of $S$ and has delay at most $Z + 1$. We prove the two directions of the equivalence separately.

$\Rightarrow$ Suppose that $Z$ can be partitioned into two sets of equal magnitude, $Z_1$ and $Z_2$. We construct a tour $T$ as follows. We start from the visibility polygon $V(p_0)$ and visit the visibility polygons of the points in $Z_1$. We then visit $V(p_0)$ again, followed by the visibility polygons corresponding to points in $Z_2$. Finally, we go back to $V(p_0)$.

The hiding cost for $p_0$ is at most $Z/2 + 2(1/6 + 1/12) = Z/2 + 1/2$. For any other point $p_i$, the hiding cost is at most $2(Z/2 + 1/3 + 1/6)$. The weighted delay for $T$ is thus

$$d_w(T) = \max \left\{ \frac{w(p_0) \cdot (Z/2 + 1/2),}{w(p_i) \cdot (Z + 1) \mid 1 \leq i \leq |Z|} \right\} \\
\leq Z + 1,$$

since $w(p_0) = 2$ and $w(p_i) = 1$ for $1 \leq i \leq |Z|$.

$\Leftarrow$ Suppose there is a tour $T$ with $d_w(T) \leq Z + 1$. By the same argument as in the proof of Theorem 1, there exists some index $j$ such that as $T$ leaves $V(p_j)$ it visits each visibility polygon of the remaining points in $S$ at least once before returning to $V(p_j)$. We call the trace that $T$ follows from $V(p_j)$ until $V(p_j)$ is reached again a cycle of $T$. We first claim that $V(p_0)$ is visited at least twice during a cycle, but this must hold, otherwise the hiding cost of $p_0$ is at least $Z$ and therefore $d_w(T) \geq 2Z > Z + 1$, giving us a contradiction. This also means that $j > 0$.

If $V(p_0)$ is visited at least three times during a cycle, we call each subpath between successive visits at $V(p_0)$ a loop and we have at least two loops completely contained in a cycle. We again have a contradiction since at least one of the loops must have horizontal width at least $1/6$ to visit the rightmost leg, and at least one of the remaining loops must have horizontal width at least $1/12$, otherwise $p_0$ has hiding cost at least $Z/2 + 1$ (and therefore $d_w(T) \geq Z + 2$ since the previous loop then must visit all the $|Z|/2$ legs with the longest length, because the legs are ordered in non-decreasing order. Any remaining loops must have horizontal width at least $\epsilon$. The point $p_1$ thus has hiding cost at least $Z + 2(1/6 + 1/12 + \epsilon + 3/12) > Z + 1$.

The visibility polygon $V(p_0)$ is therefore visited exactly twice during a cycle. Consider a subpath $X$ of $T$ between these two successive visits of $V(p_0)$. Let $X^\ast = \arg\max\{|X|, ||T \setminus X||\}$. Since $h_{c_T}(p_0) \leq Z/2 + 1/2$, otherwise $d_w(T) > Z + 1$, the length $||X^\ast|| \leq Z/2 + 1/2$ and we can let $Z^\ast$ consist of those integers in $Z$ corresponding to visibility polygons $V(p_i)$, $1 \leq i \leq |Z|$, that $X^\ast$ visits. The sum $\sum_{a \in Z} a = Z/2$ since all values in $Z^\ast$ are integral. By the same token $\sum_{a \in Z \setminus Z^\ast} a = Z/2$, since $||T \setminus X^\ast|| \leq ||X^\ast||$. Concluding the proof.

We note from the proof above that WDMSR remains NP-hard if the weights are limited to be 1 and 2.

5 Approximations

We start by showing a simple relationship between our two discrete problems.

**Theorem 3** Let $W$ be the shortest tour in $P$ that sees all points of $S$. The tour $W$ is a $2w_{\max}$-approximation to WDMSR in $P$ where $w_{\max}$ is the maximum weight of the points in $S$.

**Proof.** From Theorem 1, the tour $W$ is a 2-approximation for WDMSR when all the weights of the points are 1. If we change the weight of the point with the longest hiding cost to $w_{\max}$, we get an upper bound on the cost of the solution for the maximum weight of $w_{\max}$. It follows that the maximum weight when using $W$ is at most 2 times this weight and thus a $2w_{\max}$-approximation.

5.1 Two Weight Values

We study the case of WDMSR where points have one of two possible associated weight values, 1 and $w > 1$. From Theorem 2, we know that this restricted case is NP-hard.

We abuse language somewhat and say that a tour visits a point $p \in S$, when we actually mean that the tour intersects $V(p)$. A point $p \in S$ is called low if $w(p) = 1$ and high if $w(p) = w > 1$.

Let $W$ be the shortest tour that visits all points in $S$, let $W_L$ be the shortest tour that visits all low points in $S$, and let $W_H$ be the shortest tour that visits all the high points in $S$. Each of these tours can be computed in $O(|S|^3 n \log n)$ time. Let $m$ be the number of low points in $S$ and we construct a set of tours $U_0, \ldots, U_m$ that each visits all the points in $S$ and at least one of them has short delay.

$U_m$ is constructed as follows: let $p_x$ be some high point, follow $W_H$ from a point in $V(p_x)$ until all high
points have been visited, then move to a low point, go back to visit \( p_z \), follow \( W_w \) around again, move to another low point, go back to visit \( p_z \), etc., until all low points have been visited. \( U_m \) makes as many rounds around \( W_w \) as there are low points. Let \( d \) be the length of the path along \( U_m \) between \( V(p_z) \) and the furthest visibility polygon of any low point and let \( p_z \in \mathcal{S} \) be this low point.

The tour \( U_k \), for \( 1 \leq k \leq m \), is now constructed as follows: let \( D_k = \max \{ d, ||W_w||/2, ||W_i||/k \} \) and follow \( W_w \) from a point in \( V(p_z) \) along \( W_w \) until all high points have been visited, visit \( p_z \) and move a distance of at most \( D_k \) along \( W_i \) visiting low points before going back to \( V(p_z) \), make a tour of \( W_w \), move to \( W_i \) at the point where we left previously and move to at most \( D_k \) along \( W_i \) before going back to \( V(p_z) \), repeating until all low points have been visited. For \( k = 1 \), \( U_1 \) makes one tour around \( W_w \) and one tour around \( W_i \). Finally, we let \( U_0 = W \).

We call a tour \textit{weight separated} if every low point is visited exactly once and it can be partitioned into consecutive subpaths \( X_1, \ldots, X_n \), such that \( X_{2i+1} \) visits all high points exactly once but no low points and \( X_{2i} \) visits at least one low point but no high point. Tours \( U_1, \ldots, U_m \) are all weight separated.

\textbf{Theorem 4} At least one of the tours \( U_0, \ldots, U_m \) is a 12-approximation to the two-weight WDMSR problem in a polygon \( P \) given a finite set \( \mathcal{S} \) of points to guard.

\textbf{Proof (sketch).} Let \( T \) be an optimal solution for WDMSR in \( P \) that sees all points in \( \mathcal{S} \) and has minimum weighted delay. Pick a point on \( T \) and follow \( T \) in counterclockwise order until the traced path finishing at point \( q \) has visited each point in \( \mathcal{S} \). Let \( p_j \) be the last point of \( S \) seen by the trace. Follow \( T \) again from \( q \) in clockwise order until \( p_j \) is visited again at \( q' \) and denote this subpath by \( X \). If \( w(p_j) = w \), then \( d_w(U_0) \leq 2d_w(T) \), since \( d_w(T) \geq ||X|| \geq ||W||/2 \geq d_w(U_0)/2 \) and the path \( X \) sees all points of \( S \). Also, if \( ||W||/||W_w|| \leq 6 \), then \( d_w(U_0) \leq 12d_w(T) \). We therefore assume from now on that \( w(p_j) = 1 \) and that \( ||W_w|| < ||W||/6 \).

Let \( p_j' \) be the last point in \( S \) visited, moving on \( T \) from \( q \) to \( q' \) in clockwise order, strictly before \( q' \) and let \( q'' \) be the first point of encounter with \( V(p_j') \) (on this occasion, \( p_j' \) may have been repeatedly visited before). Let \( Y \) be this subpath of \( T \) from \( q \) to \( q'' \) and let \( T'' \) be the tour obtained by following \( Y \) from \( q \) to \( q'' \) and back to \( q \) along the same path. We ensure that \( T'' \) does not make more than one detour per low point by shortcutting any repetitions if possible. Since \( ||Y|| \leq h_{ct}(p_j) \leq d_w(T) \), we have \( h_{ct}(p_j) \leq 2||Y|| \leq 2d_w(T) \) and \( p_j \) is visited only once on \( T'' \) so this hiding cost holds for every low point. Let \( K \) be the fewest number of repetitions of any high points along \( T'' \). Again, by shortcutting any repetitions of high points, if possible, we ensure that every high point is visited exactly \( K \) times by \( T'' \) maintaining \( d_w(T'') \leq 2d_w(T) \) and that each of the \( K \) sequences of high points makes a detour to visit each high point at most once.

We transform \( T'' \) to a weight separated tour \( T''' \) as follows: find the shortest subpath \( H_1 \) of \( T'' \) that visits all high points, then follow \( T'' \) further from \( H_1 \) until a high point is reached again. We let this subpath be \( L_1 \). We continue along \( T'' \) until all high points have been visited again, giving \( H_2 \) followed by \( L_2 \) and continue subdividing \( T'' \) into \( 2K \) subpaths, \( H_1, L_1, \ldots, H_K, L_K \), each \( H_i \) visiting all the high points and \( L_i \) only visiting low points. Follow each path \( H_i \) shortcutting detours made to all low points not directly on subpaths between high points, until every high point has been visited, then go back and visit all (unvisited) low points that were shortcutted between the first and last high point and connect to \( L_i \), giving a new path \( Z_i \); see Figure 3 where high points are blue and low points are red. We have \( T'' = \bigcup_{1 \leq i \leq K} Z_i \) and \( ||H_i|| + ||L_i|| \leq ||Z_i|| \leq 3||H_i|| + ||L_i|| \).

Choose \( K^* \) to be the smallest of the two values \( K \), defined above, and \( K' \), the largest integer such that \( ||W_i||/K' > \max \{ d, ||W_w||/2 \} \). The tour \( U_{K^*} \) has hiding cost \( ||W_w|| + ||W_i||/K^* + d < 2||Z_i|| \) for each high point and each \( 1 \leq i \leq K^* \), and hiding cost \( K'||W_w|| + ||W_i|| + K'd < 2 \sum_{1 \leq i \leq K^*} ||Z_i|| \leq 2||T''|| \) for each low point, giving us \( d_w(U_{K^*}) \leq 2d_w(T'') \). Hence, \( d_w(U_{K^*}) \leq 12d_w(T) \).

\textbf{Acknowledgements}

The authors wish to thank Prof. Elmar Langetepe for fruitful initial discussions on these problems.

\textbf{References}


